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CLOSED FORM SOLUTION FOR THRESHOLD VELOCITY FOR INITIATION OF SEDIMENT MOTION UNDER WAVES

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Abstract: The present paper presents a rational expression for the initiation of motion for non-cohesive sediment under rough turbulent wave conditions. Previously proposed relationships for the initiation of motion are examined as well as laboratory measurements corresponding to fully rough turbulent boundary layers (which is normally the case under prototype conditions). By combining the modified Shields criterion as proposed by Bagnold (1963) with the wave friction factor, f_w, from Jonsson (1966), an easily applied criterion for the threshold velocity for incipient motion under rough turbulent conditions. The new relationship is formulated as $U_{w,cr} = 4\sqrt{\varphi_{cr}}\left[1.1\ \log_{10}\left(\sqrt{\varphi_{cr}T}/(\pi d_{50})\right) - 0.08\right]^{1/0.9}$. Predictions with the new criterion are shown to be in good agreement with measurements over the full range of typically encountered engineering conditions.

INTRODUCTION

The threshold of sediment motion is of central concern in calculating littoral transport under wave action. Based on experimental data in steady current, Shields (1936) proposed a critical value for the bottom shear stress as a function of the grain Reynolds number $\Re_* = u_* d_{50}/\nu$ where $u_* = \sqrt{\tau/\rho}$ is the friction velocity, τ is the bottom shear stress, ρ is the fluid density, ν the kinematic viscosity of the fluid, and d_{50} is the median grain size. He proposed to use a dimensionless shear stress to characterize the threshold value for the sediment transport. This dimensionless shear stress is nowadays well-known as the Shields parameter:

$$\theta_{i,cr} = \frac{f_i U_i^2}{2(s-1) g d_{50}} \tag{1}$$

where the subscript i may be replaced by c (current related terms) or w (wave related terms), f is the friction factor, U is the horizontal near-bottom velocity, $s = \rho_s/\rho$ is the relative density of the sediment, and g the acceleration due to gravity. He found, using steady current data (*c.f.* Fig. 1), that the critical Shields parameter varies from 0.04 to 0.07 depending on the particular Reynolds number.

For practical applications, the Shields relationship is, however, inconvenient as the critical shear velocity appears on both sides. For this reason, many authors (Valembois 1960; Madsen and Grant 1976; van Rijn 1984; Soulsby 1997) proposed relationships between the critical Shields parameter and the dimensionless grain size $d_* = (g(s-1)/\nu^2)^{1/3} d_{50}$ (see Fig. 1). A simple algebraic expression that fits Shields' curve closely was proposed by Soulsby and Whitehouse (1997):

$$\theta_{c,cr} = \frac{0.24}{d_*} + 0.055\left[1 - \exp(-0.02 d_*)\right] \tag{2}$$

However, using more recent data, Soulsby (1997) observed much smaller values for $\theta_{c,cr}$ when $d_* < 2$. He thus proposed a modification of Eq. 2(Fig. 1):

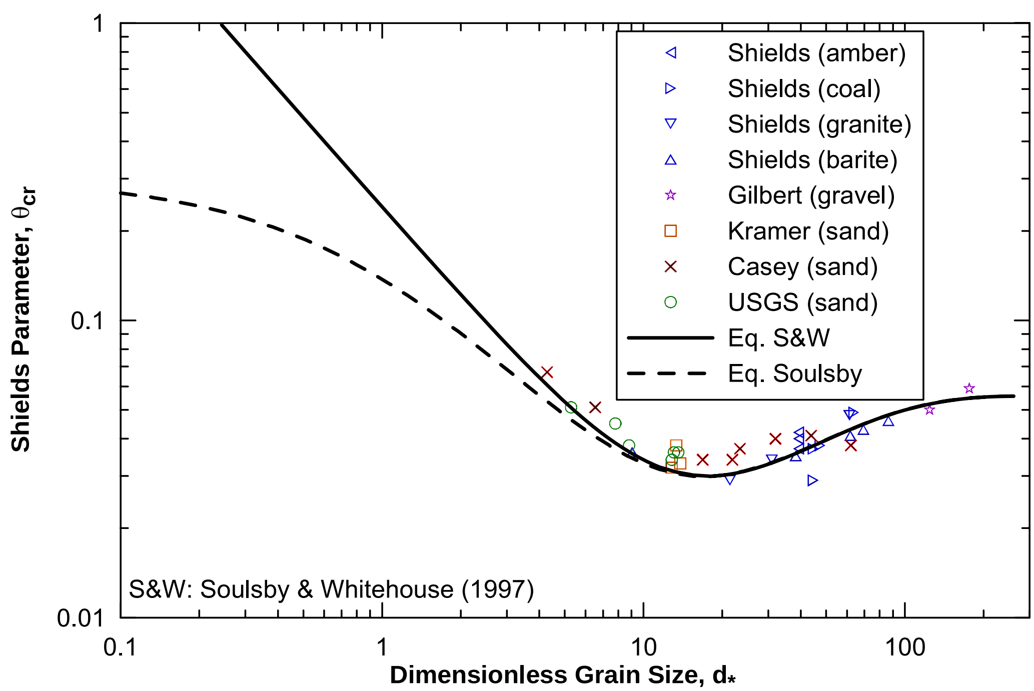

Fig. 1. Critical Shields parameter as a function of the dimensionless grain size for steady current data (data from Brownlie 1981).

$$\theta_{c,cr} = \frac{0.30}{1+1.2d_*} + 0.055\left[1-\exp\left(-0.02d_*\right)\right] \quad (3)$$

It is, however, difficult to define the exact threshold of sediment transport from the incipient motion regime where a few particles on the bed surface are seen to be moving and the general surface motion which corresponds to a well-defined and reproducible transition from occasional motion to general motion of the uppermost particles (Chan *et al.* 1972).

Many authors (van Rijn 1984; Soulsby 1997) also assumed that the results for steady currents can be extended to waves, and combined waves and currents using the maximum value of the Shields parameter over a wave cycle. However, large uncertainties about the critical Shields parameter seem to occur in the case of waves as the estimation of the wave-related friction coefficient strongly affects the results. As Camenen and Larson (2005) pointed out, the estimation of the critical Shields parameter has a significant influence on the bed load flux for low regimes. It is, thus, fundamental to provide better estimation of the threshold velocity and the critical Shields parameter for the inception of movement. The main objective of this study is to derive a closed form solution for the threshold velocity for initiation of sediment motion under oscillatory water waves that has a reasonable accuracy and that is easy to use.

ESTIMATION OF THE CRITICAL VELOCITY FOR THE INITIATION OF SEDIMENT MOVEMENT

Background

A large number of studies (mostly in laboratory) and empirical relationships concerning the threshold velocity for sediment motion have been presented in the literature. For example, Madsen and Grant (1976) demonstrated that the critical Shields parameter together with the non-dimensional parameter $S_* = \sqrt{(s-1)gd_{50}^3}/(4\nu)$, give a good description of reanalyzed older experimental data, provided that the boundary shear stress is related to the wave friction factor, f_w, as presented in Jonsson (1966). Like the ordinary Shields curve, this modified version is somewhat difficult to use in a practical situation. Here, the Shields parameter is based on the maximum value of the boundary shear stress during one wave period. Komar and Miller (1975) and Komar (1976) found, also using the results of Jonsson (1966), that the threshold condition is well described by:

$$\frac{U_{w,cr}^2}{(s-1)gd_{50}} = \alpha\left(\frac{2A_{w,cr}}{d_{50}}\right)^{\beta} \quad (4)$$

where $U_{w,cr}$ is the maximum near-bottom horizontal velocity and $A_{w,cr}$ is the water particle horizontal amplitude at the bottom according to linear wave theory. For grain sizes smaller than 0.5 mm, α =0.21 and β = 0.5, respectively; and for grain sizes greater than 0.5 mm the coefficient values are set to α =0.68π and β =0.25, respectively. These two expressions are comprehensive and easy to use. There is, however, a discrepancy between the two expressions in the region of small wave periods and grain diameters around 0.5 mm. According to the authors themselves, the diagram should not be used in this area. Thus, it is not valid for waves and sediment typical for a large number of beaches around the world. Hallermeier (1980) proposed a very simple criterion where the threshold velocity only depends on sediment diameter and density according to:

$$U_{w,cr} = \sqrt{8(s-1)gd_{50}} \tag{5}$$

In the present study it will be shown, after comparison with experimental data, that this relationship is too simplistic. In conclusion, for the determination of threshold conditions for initiation of sediment motion, there seems to be a choice between more "accurate" curves which are somewhat complicated to use and more convenient curves which are not valid under typical field conditions.

Chan *et al.* (1972) investigated the effect of the kinematic viscosity (no significant influence found) and the relative particle density on the inception of the sediment transport (general movement of the upper layer). Using their own experimental data with various particles characteristics and fluid densities, they proposed the following empirical relationship, where T is the wave period:

$$U_{w,cr} = 0.37\left(\frac{2\pi}{T}\right)^{-0.5}\left[(s-1)g\right]^{0.75} d_{50}^{0.25} \tag{6}$$

Proposed threshold relationship

Employing the classical definition of the wave friction factor, f_w, according to $\tau_{w,cr} = 0.5 f_w U_{w,cr}^2$ together with the definition of the Shields parameter, $\theta_{w,cr}$ (cf. Eq. 1), we can derive:

$$f_w U_{w,cr}^2 = 2\theta_{w,cr}(s-1)gd_{50} \equiv \varphi_{cr} \tag{7}$$

In the rough turbulent case, the wave friction factor, f_w, is independent of the wave Reynold's number and related to the relative roughness k_s/A_w only (Jonsson 1966) as

$$\left(\frac{1}{4\sqrt{f_w}}\right) + \log_{10}\left(\frac{1}{4\sqrt{f_w}}\right) = \log_{10}\left(\frac{A_w}{k_s}\right) - 0.08 \tag{8}$$

where k_s is the Nikuradse roughness height. It is difficult to estimate k_s for a rippled and duned bottom, but for a plane horizontal bottom of non-cohesive material, it may be taken as $k_s = 2d_{50}$ (Yalin 1977). It should be noted however that Eq. 8 is not applicable for small values of A_w/k_s. In order to better fit the data (see below), the original formulation of (Jonsson 1966) is slightly modified in this study according to:

$$\left(\frac{1}{4\sqrt{f_w}}\right)^{0.9} + \log_{10}\left(\frac{1}{4\sqrt{f_w}}\right)^{1.1} = \log_{10}\left(\frac{A_w}{k_s}\right)^{1.1} - 0.08 \qquad (9)$$

With $A_w = U_w T/(2\pi)$, Eq. (9) may be combined with Eq. (7) to yield a closed form solution of the critical threshold velocity as:

$$U_{w,cr} = 4\sqrt{\varphi_{cr}}\left[1.1\ \log_{10}\left(\frac{\sqrt{\varphi_{cr}}T}{\pi d_{50}}\right) - 0.08\right]^{1/0.9} \qquad (10)$$

In order to evaluate this proposed Eq. (10), comparisons need to be made against experimental results. These will be made in three stages as the threshold velocity depends on the determination of the critical Shields parameter value as well as the wave friction factor.

COMPARISON AGAINST EXPERIMENTAL DATA

Wave Friction Factor

For the wave friction factor, Eq. (9) is plotted against the available data together with the proposed formulations of Jonsson (1966), Swart (1974), and Nielsen (1992) in Fig. 2. As seen from Table 1 (where *e.g.* $P_{1.2}$ means percentage of calculated values within a factor of 1.2 from the measurements) Eq. (9) and the Nielsen (1992) relationship appear to yield the best results among the studied formulas.

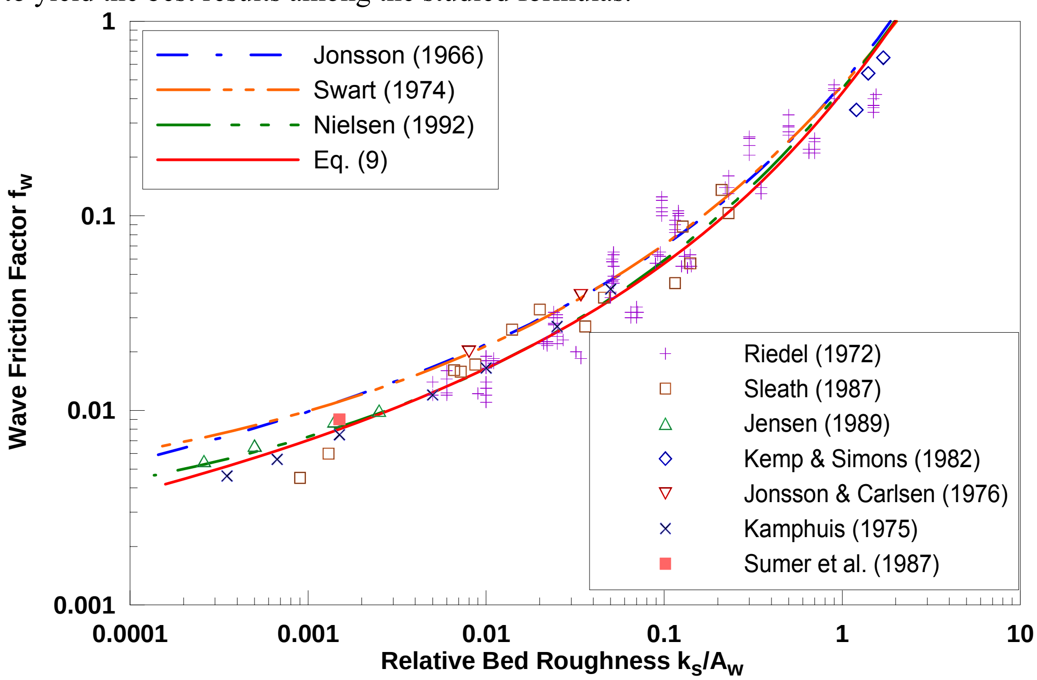

Fig. 2. Proposed relationships and measured values for the wave friction factor in the rough turbulent regime.

Table 1. Calculated measures of accuracy (%) of the calculated wave friction factor.			
	$P_{1.2}$	$P_{1.5}$	P_2
Jonsson (1966)	39	74	98
Swart (1974)	35	74	98
Nielsen (1992)	42	83	98
Eq. (9)	44	83	98

Critical Shields Parameter

As previously discussed, the Shields criterion may be extended to define the threshold of sediment motion under oscillatory water waves. Fig. 3 shows threshold data under waves together with the proposed expressions for a steady current (Eqs. 2 and 3). It is obvious that the data do not support the lines derived from the original Shields curve. Based on this result, two new expressions for the critical Shields parameter are proposed here. As opposed to the previous estimation based on steady current, it seems that $\theta_{w,cr}$ is not increasing with smaller values of d_* when $d_* < 10$ or for larger values of d_* when d_* >100. Also, it seems that larger values of $\theta_{w,cr}$ are observed for larger values of d_* under waves compared to the results for steady currents. As a straight-forward approach, a representative value on the critical Shields parameter of $\theta_{w,cr} = 0.07$ (based on a best fit) independent of grain size is suggested (dash-dot line in Fig. 3). One would suspect, however, that the critical Shields parameter for the inception of movement should be a function of the dimensionless grain size (*cf.* Eq. 5). For this reason, a simple relationship dependent on grain size (full line in Fig. 3) is also proposed for testing against the data:

$$\theta_{w,cr} = 0.08\left[1 - \exp\left(-15/d_* - 0.02d_*\right)\right] \tag{11}$$

Table 2 examines the accuracy of the predictions within specified limits for the different relationships together with the average value and standard deviation of the difference $\Delta U_{w,cr} = U_{w,cr,c} - U_{w,cr,m}$ where $U_{w,cr,c}$ and $U_{w,cr,m}$. are the calculated and measured values of the critical Shield parameter values, respectively. The *P*-values are a direct measure of the accuracy of the predictive relationship. A small avg($\Delta U_{w,cr}$) value indicates that the predicted values are evenly spread on either sides of the correct value while a small std($\Delta U_{w,cr}$) value indicates that the predicted values are close together. None of these two latter values are, however, a direct measure of accuracy.

Table 2. Calculated measures of accuracy (%) and spreading of the calculated critical Shields parameter.					
	$P_{1.2}$	$P_{1.5}$	P_2	avg($\Delta U_{w,cr}$)	std($\Delta U_{w,cr}$)
Soulsby & Whitehouse (1997)	19	51	86	-0.0200	0.0152
Soulsby (1997)	19	49	82	-0.0208	0.0159
$\theta_{w,cr} = 0.07$	34	73	94	0.0114	0.0171
Eq. (11)	56	88	99	0.0045	0.0137

As seen from Table 2, Eq. (11) significantly improves the results compared to the other relationships. This equation is quite similar in its form to that of Eq. (2) (Soulsby and Whitehouse 1997) except that it does not grow for smaller values of d_*. The minimum

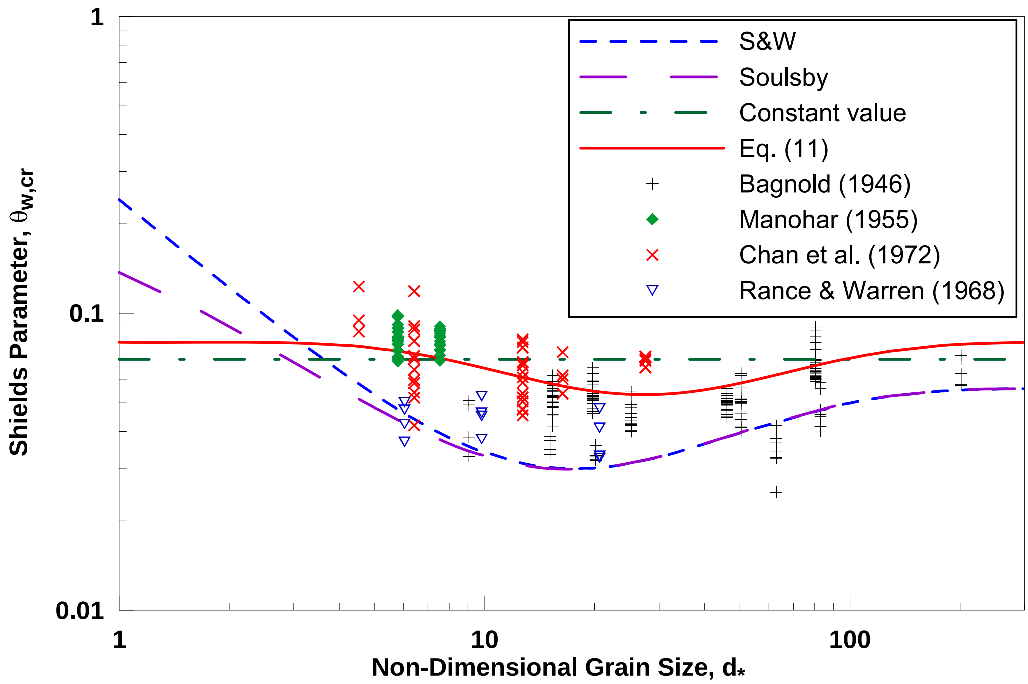

Figure 3: Critical Shields parameter as a function of the dimensionless grain size under oscillatory water waves.

value for $\theta_{w,cr}$ is also observed for a larger d_* ($d_{*,min}$ = 40 instead of 20 looking at the Shields study). Soulsby (1997) suggested that the average rather than the peak bed shear stress may be used in case of waves (and wave and current interaction). Using Eq. (3), this would, however, lead to an underestimation of the results for the critical Shields parameter prediction The results for the fixed Shields parameter value is doing surprisingly well, much better that Soulsby and Whitehouse (1997) or Soulsby (1997). While the constant value has no dependency on grain size, the other two seem to be too sensitive to the grain size. It appears, thus, that the results for a steady current cannot so easily be extended to situations involving waves.

Critical Velocity for Initiation of Motion

In order to compare Eq. (10) with experimental results, portions of the data from Bagnold (1946), Manohar (1955) and Rance and Warren (1968), corresponding to fully rough turbulent conditions, as well as the data from Chan *et al.* (1972) were examined. Measured values of the incipient velocities $U_{w,cr,m}$ are compared to those given by Eq. (10) $U_{w,cr,c}$, where Shields parameter is given as a constant and by Eq. (11), respectively (*cf.* Tab. 3). In addition, the proposed solution is compared to those of Chan *et al.* (1972), Komar and Miller (1975), and Hallermeier (1980), respectively. Statistically, the proposed Eq. (10) using a fixed Shields parameter doing somewhat better than that of Komar and Miller and significantly better than that of Chan *et al.* or Hallermeier. Eq. (10) in combination with a varying Shields parameter according to Eq. (11) is not strong on the $P_{1.2}$ level but doing much better on the other two levels.

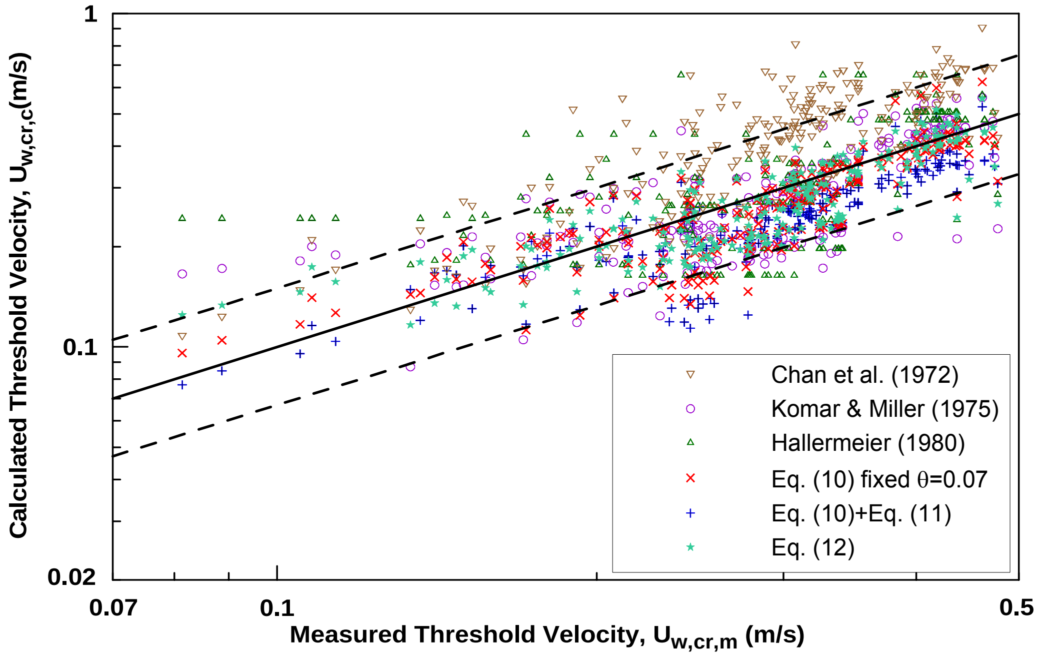

Figure 4: Calculated vs. measured threshold velocities for the initiation of motion under oscillatory water waves. Solid line indicates calculated value = measured value. Dashed lines indicate values within a factor of 1.5.

Following the work by Chan *et al*. (1972) and Hallermeier (1980) but using all the available data, a simple empirical function for the threshold velocity is proposed (see Eq. (12) in Fig. 4 and Table 3):

$$U_{w,cr} = 0.5\sqrt{\left[(s-1)g\right]^{1.25} d_{50}^{0.75} T^{0.5}} \tag{12}$$

The comparison against the experimental data shows that Eq. (12) yields better results than even Eq. (10) with a fixed Shields parameter. Also, it yields much better results compared to the other formulas.

Table 3. Calculated measures of accuracy (%)and spreading of the calculated threshold velocity for initiation of motion under oscillatory water waves.

	$P_{1.2}$	$P_{1.5}$	P_2	avg($\Delta U_{w,cr}$)	std($\Delta U_{w,cr}$)
Chan et al. (1972)	28	62	95	0.126	0.107
Komar & Miller (1975)	62	84	99	-0.010	0.067
Hallermeier (1980)	50	76	95	0.021	0.097
Eq. (10) fixed θ= 0.07	65	91	100	-0.024	0.052
Eq. (10) + Eq. (11)	45	93	98	-0.050	0.046
Eq. (12)	62	97	100	-0.020	0.049

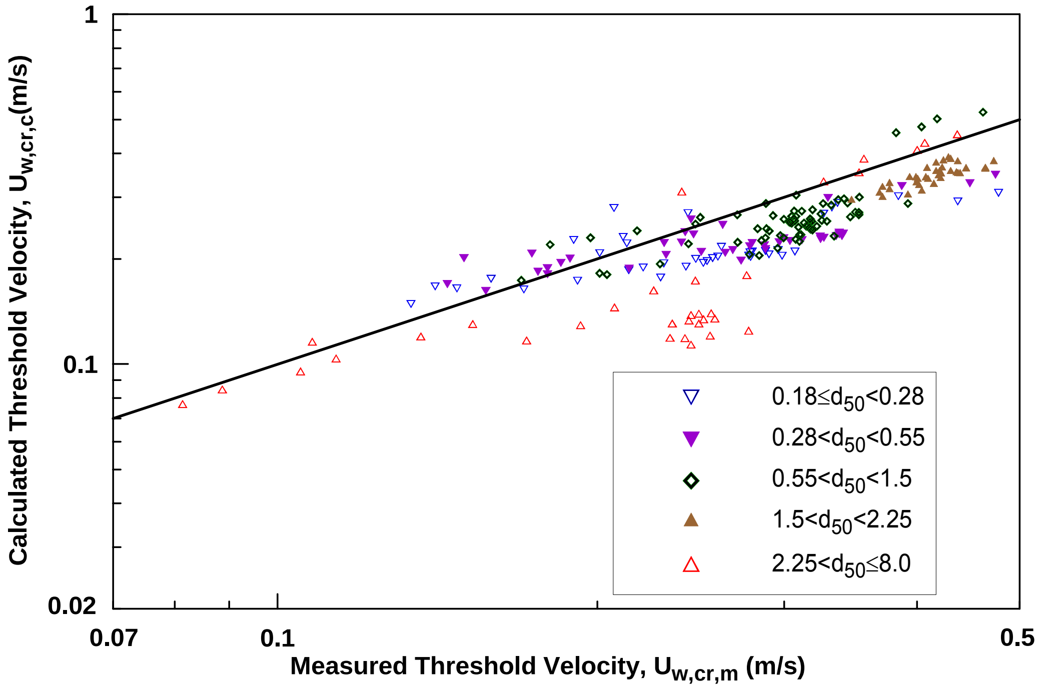

Fig. 5. Estimation of the threshold velocity; comparison between Eqs. (13)-(15) and experimental data. The straight line indicates calculated value = measured value.

Recommended Relationships

From these results it would be tempting to recommend Eq. (12). However, as the threshold velocity and the critical Shields parameter are linked it seems more appropriate to use a coherent set of $\theta_{w,cr}$ and $U_{w,cr,m}$ that both are in reasonable agreement with data.

Thus, this study leads to the recommendation to use Eq. (11) for $\theta_{w,cr}$ and then use Eq. (10) in combination with Eq. (11) to calculate $U_{w,cr,m}$, *i.e.*

$$\theta_{w,cr} = 0.08\left[1 - \exp\left(-15/d_* - 0.02d_*\right)\right] \tag{13}$$

$$U_{w,cr} = 4\sqrt{\varphi_{cr}}\left[1.1\ \log_{10}\left(\frac{\sqrt{\varphi_{cr}}T}{\pi d_{50}}\right) - 0.08\right]^{1/0.9} \tag{14}$$

with

$$\varphi_{cr} = 2\theta_{w,cr}\left(s-1\right)gd_{50} \tag{15}$$

In Fig. 5, the threshold velocity $U_{w,cr}$, estimated from Eqs. (13)-(15) is compared to experimental measurements. It shows the very good correlation between the theory and the data. It seems, however, that the predictions are slightly underestimated for all grain sizes, especially for the coarser particles.

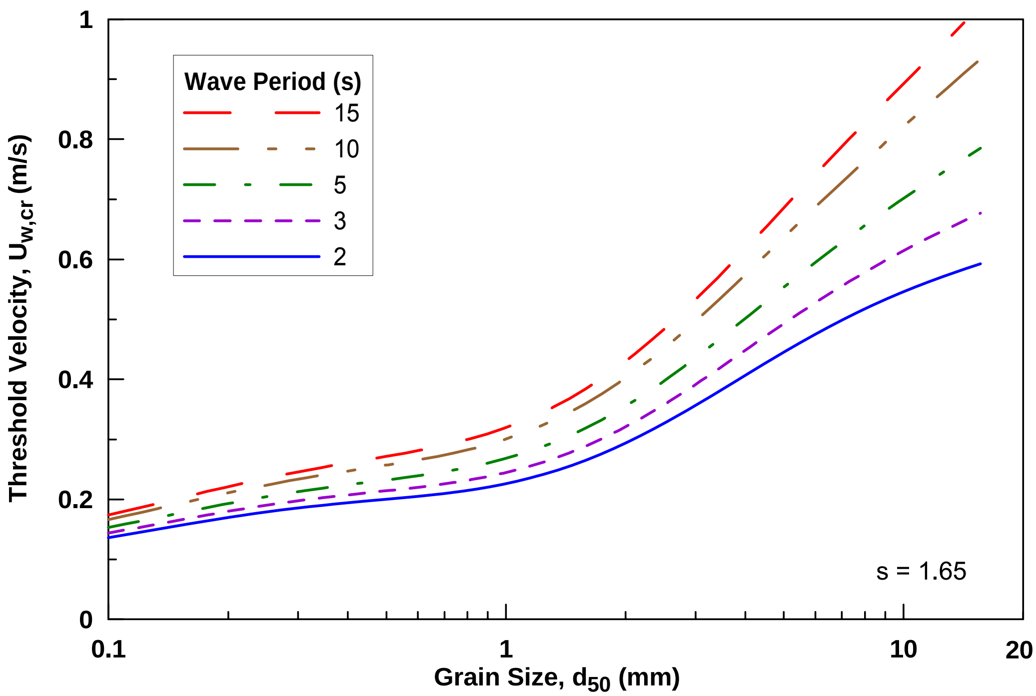

Fig. 6. Proposed near-bottom velocities for initiation of motion under rough, turbulent oscillatory water waves.

Having gained confidence in Eq. (14) it is now possible to draw proposed "design curves" for the threshold velocity for initiation of motion as shown in Fig. 6. For smaller grain sizes, the influence of wave period is less pronounced than for coarser sediments.

Critical Wave Height for Initiation of Motion

With the critical bottom velocity determined by Eq. (10), the corresponding critical wave height H_{cr} at a particular depth h_{cr} may be calculated according to linear wave theory as:

$$H_{cr} = \frac{U_{w,cr} T}{\pi} \sinh\left(\frac{2\pi h_{cr}}{L}\right) \tag{13}$$

where L = wave length at depth h_{cr}. This relation is displayed in non-dimensional form in Fig.7. Inversely, this relationship may be solved for the water depth h_{cr} out to which sediment will move under a specific wave condition to yield:

$$h_{cr} = \frac{L}{2\pi} \ln\left(\frac{\pi H_{cr}}{U_{w,cr} T} + \sqrt{\left(\frac{\pi H_{cr}}{U_{w,cr} T}\right)^2 + 1}\right) \tag{14}$$

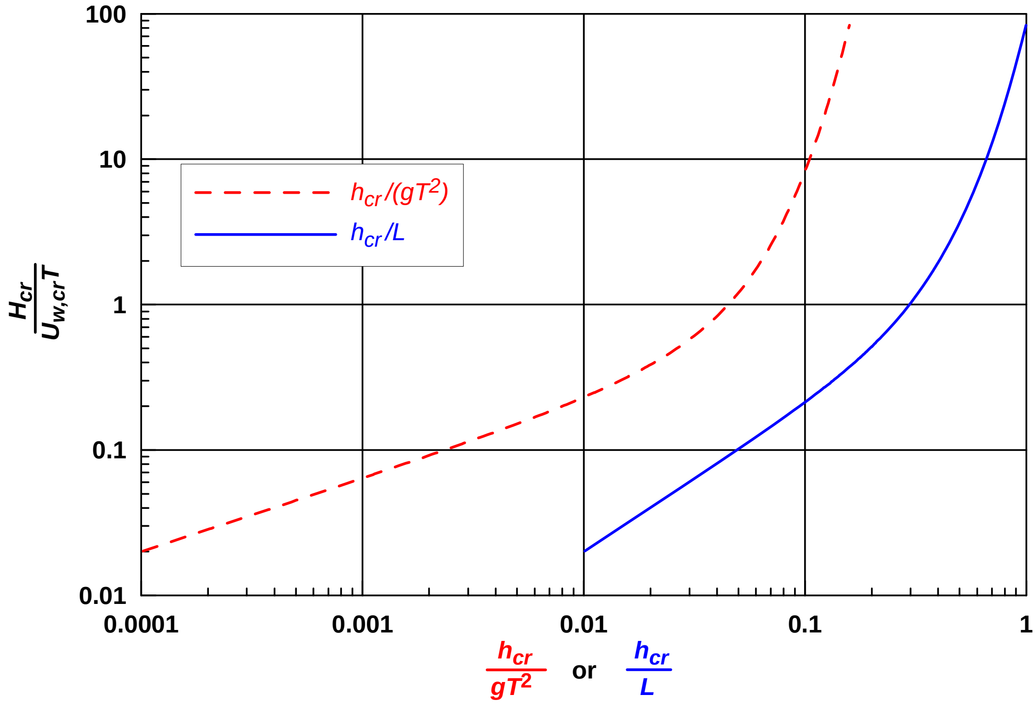

Fig. 7. Non-dimensional wave height $H_c/(U_{w,cr}T)$ versus non-dimensional water depth $h_c/(gT^2)$for the initiation of motion under rough, turbulent oscillatory water waves.

CONCLUSIONS

A study of the threshold velocity and critical Shields parameter has been undertaken for rough oscillatory flows. New relationships were proposed for the estimation of the critical Shields parameter as well as the threshold velocity for initiation of sedment motion under waves (Eqs. 13-15). The expression for the threshold velocity is an easy to use criterion for practical evaluation of sediment threshold conditions under rough, turbulent field wave conditions. The proposed relation has no ambiguity for any applicable grain size or wave period.

The proposed relationship for the threshold velocity has a tendency to underpredict measured threshold velocities, especially for coarser grain sizes. These discrepancies are mainly caused by factors inherent in this simplified derivation but also due to the difficulties in identifying the threshold situation in the laboratory. However, threshold velocities in nature are expected, as a result of sloping and rippled bottom, to be lower than those measured for plane horizontal bottom in the laboratory.

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